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A characterization of extended monotone metrics

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ABSTRACT

Classical information geometry has emerged from the study of geometrical aspect of the statistical estimation. Cencov characterized the Fisher metric as a canonical metric on probability simplexes $S_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum x_i = 1\}$ from a statistical point of view, and Campbell extended the characterization of the Fisher metric from probability simplexes to positive cone $\mathbb{R}^n_+$. In quantum information geometry, quantum states which are represented by positive Hermitian matrices with trace one are regarded as an extension of probability distributions. A quantum version of the Fisher metric is introduced, and is called a monotone metric. Petz characterized the monotone metrics on the space of all quantum states in terms of operator monotone functions. A purpose of the present paper is to extend a characterization of monotone metrics from the space of all states to the space of all positive Hermitian matrices on finite dimensiona l Hilbert space. This characterization corresponds quantum modification of Campbell's work.

1. Introduction

Classical information geometry, which was initiated by Amari [1,2], Amari and Nagaoka [3], has emarged from the study of geometrical aspects of the statistical estimations, and nowadays it offers geometrical tools in statistics and information theory (see [4]). It is well known that the Fisher metric is introduced by C.R. Rao as a Riemannian metric and plays an important role in classical information geometry. The Fisher metric is characterized as the Riemannian metric which has monotonicity under Markov mappings on the probability simplexes $S_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum x_i = 1\}$ [6]. Campbell [5] extended the above characterization from probability simplexes to positive cones $\mathbb{R}^n_+$, and gave a geometrical interpretation for some statistical concepts.

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Quantum information geometry is motivated by an application of information geometry in quantum states. A state space of the n-level system is defined by the set of all positive Hermitian $n \times n$ matrices of trace one. Since a quantum state space includes probability simplexes as diagonal matrices, it is regarded as an extension notion of probability simplexes. Then it is natural to consider a quantum version of the Fisher metric when we deal with statistics and information theory in quantum settings [9]. Since the Fisher metric on probability simplexes is characterized by the monotonicity with respect to Markov mappings, a quantum version of the Fisher metric is defined by the monotonicity with respect to a quantum version of Markov mappings which are called TP-CP mappings and is called a monotone metric respectively [8]. A characterization of monotone metrics in terms of operator monotone functions was given by Petz [10].

The purpose of this paper is to develop a characterization theorem which is closely related to Petz's. The difference between ours and Petz's is that we characterize the metrics on the positive Hermitian matrices H_n^+, while Petz characterizes them on the quantum state spaces $\mathcal{M}_{n-1} = \{\rho \in H_n^+ \mid Tr\rho = 1\}$. When we regard $\mathcal{M}_{n-1}$ as a submanifold of H_n^+, our theorem includes the Petz's characterization. This characterization corresponds quantum modification of Campbell's work.

2. Notation and definition

We denote by $M_n(\mathbb{C})$ the set of all $n \times n$ matrices, by H_n^0 the set of all Hermitian matrices with trace zero, by H_n^+ the set of all positive Hermitian matrices, respectively.

Definition 2.1. A mapping $\mathbf{T} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is called trace preserving completely positive (TP-CP) if $\mathbf{T}(H_n^+) \subset H_m^+$, $\mathbf{T}$ preserves trace and $\mathbf{T} \otimes id_k : M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_k(\mathbb{C})$ has positivity for any $k \in \mathbb{N}$ (i.e. $0 \leqslant \mathbf{T} \otimes id_k(A)$ for any $0 \leqslant A \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$).

Definition 2.2. $K : (A, B, \rho) \in \bigcup_{n=1}^{\infty}(M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times H_n^+) \mapsto K_\rho(A, B) \in \mathbb{C}$ is called an extended monotone metric when it satisfies the following conditions:

(a) $(A, B) \mapsto K_\rho(A, B)$ is sesquilinear for every $\rho \in H_n^+$.
(b) $K_\rho(A, A) \geqslant 0$ and the equality holds if and only if $A = 0$.
(c) $\rho \mapsto K_\rho(A, A)$ is continuous on H_n^+ for every $A \in M_n(\mathbb{C})$.
(d) $K_{\mathbf{T}(\rho)}(\mathbf{T}(A), \mathbf{T}(A)) \leqslant K_\rho(A, A)$ for every TP-CP mapping $\mathbf{T}$, $\rho \in H_n^+$ and A $\in M_n(\mathbb{C})$.

The restriction of an extended monotone metric to $\bigcup_{n=1}^{\infty}(H_n^0 \times H_n^0 \times \mathcal{M}_{n-1})$ is called a monotone metric.

Since the Definition 2.2. implies a unitary invariance $K_{U\rho U^*}(UAU^*, UBU^*) = K_\rho(A, A)$ for an arbitrary unitary matrix U, therefore ρ always can be assumed to be diagonal.

Definition 2.3. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called operator monotone if $f(A) \leqslant f(B)$ for all pairs of A, B satisfy $0 < A \leqslant B$.

3. Main theorem

Let $\langle\, , \,\rangle_{HS}$ be the Hilbert–Schmidt inner product, and set $\mathbf{L}_A(B) = AB$, $\mathbf{R}_A(B) = BA$ for $A\, B \in M_n(\mathbb{C})$, $K_\rho^f(A, B) = \langle A, (f(\mathbf{L}_\rho \mathbf{R}_\rho^{-1})\mathbf{R}_\rho)^{-1}(B)\rangle_{HS}$. A characterization of monotone metrics by Petz is the following.

Theorem 3.1 (Petz [10]). *Let $(A, B) \mapsto K_D(A, B)$ be a monotone metric. Then there exists uniquely an operator monotone function f such that $K_\rho(A, B) = K_\rho^f(A, B)$.*

The main theorem of the present paper is following.

Theorem 3.2. *There is a following one-to-one correspondence between extended monotone metrics* $K : \bigcup_{n=1}^{\infty}(M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times H_n^+) \to \mathbb{C}$ *and pairs of a continuous function* $b : \mathbb{R}_+ \to \mathbb{R}$ *and a continuous family of operator monotone functions* $\{f_t : \mathbb{R}_+ \to \mathbb{R}_+\}_{t \in \mathbb{R}_+}$ *such that* $tb(t) + f_t(1)^{-1} > 0$.

$$K_\rho(A, B) = b(Tr\rho)(TrA)^*(TrB) + K_\rho^{f_{Tr\rho}}(A, B).$$

This theorem includes Pet'z characterization in Theorem 3.1 when we regard $\mathcal{M}_{n-1}$ as a submanifold of H_n^+.

Since density matrices include probability distributions as diagonal matrices, we can translate a consequence of Campbell [5] from the classical setting into a quantum one as follows.

Lemma 3.1. *Let K be an extended monotone metric. Then there exists uniquely a pair of a continuous function* $b : \mathbb{R}_+ \to \mathbb{R}$ *and a continuous positive function* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$K_\rho(A, B) = b(Tr\rho)(TrA)^*(TrB) + (Tr\rho)c(Tr\rho)Tr(\rho^{-1}A^*B),$$

where $A, B \in M_n(\mathbb{C})$ *and* $\rho \in H_n^+$ *are mutually commutative.*

Lemma 3.2. *Let K be an extended monotone metric. Then there exists uniquely a pair of a continuous function* $b : \mathbb{R}_+ \to \mathbb{R}$ *and a continuous positive function* $d : \mathbb{R}_+^3 \to \mathbb{R}_+$ *such that*

$$tb(t) + d(1, 1; t) > 0,$$

$$d(r\lambda, r\mu; t) = r^{-1}d(\lambda, \mu; t) \quad (r, \lambda, \mu, t \in \mathbb{R}_+),$$

$$K_{\rho=diag(\lambda_1,\ldots,\lambda_n)}(A, A) = b(Tr\rho)|TrA|^2 + \sum d(\lambda_i, \lambda_j; Tr\rho)|A_{ij}|^2.$$

Proof. Unless otherwise noted, we assume $n \geqslant 2$ and ρ indicates $diag(\lambda_1, \ldots, \lambda_n) \in H_n^+$. The uniqueness of such a pair b and d is clear. We denote the $n \times n$ unit matrix and the $n \times n$ matrix unit by I_n and $E_{ij}^{(n)}$, respectively.

Set $U_i = diag\left(1, \ldots, \overset{i}{\check{\sqrt{-1}}}, \ldots, 1\right)$, and using the unitary invariance of an extended monotone metric,

$$K_\rho\left(E_{ij}^{(n)}, E_{kl}^{(n)}\right) = K_{U_i\rho U_i^*}\left(U_iE_{ij}^{(n)}U_i^*, U_iE_{kl}^{(n)}U_i^*\right) = K_\rho\left(\sqrt{-1}E_{ij}^{(n)}, E_{kl}^{(n)}\right) = \sqrt{-1}K_\rho\left(E_{ij}^{(n)}, E_{kl}^{(n)}\right)$$

where i is different from j, k, l. Therefore, we get $K_\rho\left(E_{ij}^{(n)}, E_{kl}^{(n)}\right) = 0$ where i is different from j, k, l. Similarly, we get $K_\rho\left(E_{ij}^{(n)}, E_{kl}^{(n)}\right) = 0$, where one of i, j, k, l is different from the others, and $K_\rho\left(E_{ij}^{(n)}, E_{ji}^{(n)}\right) = 0$ where $i \neq j$.

By using the above calculation, we can show that $K_\rho\left(E_{ij}^{(n)}, E_{ij}^{(n)}\right)$ $(i \neq j,$ and $n \geqslant 2)$ is determined only by λ_i, λ_j and $Tr\rho$. We denote $K_\rho(A, A)$ by $\|A\|_\rho^2$. Set

$$\mathbf{T}_n\left(E_{ij}^{(n)}\right) = \begin{cases} E_{ij}^{(3)} & \text{if } i, j = 1, 2, \\ E_{33}^{(3)} & \text{if } i = j = 3, 4, \ldots \\ 0 & \text{otherwise}. \end{cases}$$

$$\mathbf{T}_\rho\left(E_{ij}^{(3)}\right) = \begin{cases} E_{ij}^{(n)} & \text{if } i, j = 1, 2, \\ \frac{1}{Tr\rho - \lambda_1 - \lambda_2}\sum_{k=3}^{n}\lambda_k E_{kk}^{(3)} & \text{if } i = j = 3, \\ 0 & \text{otherwise}. \end{cases}$$

Since these are TP-CP,

$$\begin{aligned}\|E_{12}^{(n)}\|_\rho &\geqslant \|\mathbf{T}_n(E_{12}^{(n)})\|_{\mathbf{T}_n(\rho)}\\ &= \|E_{12}^{(3)}\|_{diag(\lambda_1,\lambda_2,Tr\rho-\lambda_1-\lambda_2)}\\ &\geqslant \|\mathbf{T}_\rho\circ\mathbf{T}_n(E_{12}^{(n)})\|_{\mathbf{T}_\rho\circ\mathbf{T}_n(\rho)}\\ &= \|E_{12}^{(n)}\|_\rho.\end{aligned}$$

This implies $\|E_{12}^{(n)}\|_\rho = \|E_{12}^{(3)}\|_{diag(\lambda_1,\lambda_2,Tr\rho-\lambda_1-\lambda_2)}$. The matrix U_{ij} represents the unitary matrix which exchanges the ith coordinate for the 1st coordinate, jth coordinate for the 2nd coordinate, and does not change other coordinates. Then, we get

$$\|E_{ij}^{(n)}\|_\rho = \|U_{ij}E_{ij}^{(n)}{U_{ij}}^*\|_{U_{ij}\rho {U_{ij}}^*} = \|E_{12}^{(n)}\|_{diag(\lambda_i,\lambda_j,\dots)} = \|E_{12}^{(3)}\|_{diag(\lambda_i,\lambda_j,Tr\rho-\lambda_i-\lambda_j)}.$$

We define $d(\lambda,\mu;t) := \|E_{12}^{(3)}\|^2_{diag(\lambda,\mu,t-\lambda-\mu)}(0<\lambda,\mu \text{ and } \lambda+\mu<t)$. Then it holds

$$d(r\lambda,r\mu;t) = r^{-1}d(\lambda,\mu;t)\quad (0<r,\lambda,\mu \text{ and } \lambda+\mu, r\lambda+r\mu<t).$$

We show the above equality. For an arbitrary natural number $m \geqslant 1$,

$$\begin{aligned}d(\lambda,\mu;t) &= \|E_{12}^{(3)}\|^2_{\rho=diag(\lambda,\mu,t-\lambda-\mu)}\\ &= \left\|E_{12}^{(3)}\otimes\frac{1}{m}I_m\right\|^2_{\rho\otimes\frac{1}{m}I_m}\\ &= \frac{1}{m^2}\sum_{i,j}^{m} K_{\rho\otimes\frac{1}{m}I_m}(E_{12}^{(3)}\otimes E_{ii}^{(m)}, E_{12}^{(3)}\otimes E_{jj}^{(m)})\\ &= \frac{1}{m^2}\sum_{i=1}^{m}\|E_{12}^{(3)}\otimes E_{ii}^{(m)}\|^2_{\rho\otimes\frac{1}{m}I_m}\\ &= \frac{1}{m}\|E_{12}^{(3)}\otimes E_{11}^{(m)}\|_{\rho\otimes\frac{1}{m}I_m}\\ &= \frac{1}{m}d\left(\frac{\lambda}{m},\frac{\mu}{m};t\right).\end{aligned}$$

Thus, for an arbitrary positive rational number $q=\frac{l}{m}(1\leqslant l, m$ are natural numbers),

$$\begin{aligned}d(q\lambda,q\mu;t) &= d\left(l\frac{\lambda}{m}, l\frac{\mu}{m};t\right)\\ &= \frac{1}{l}d\left(\frac{\lambda}{m},\frac{\mu}{m};t\right)\\ &= \frac{m}{l}d(\lambda,\mu;t)\\ &= q^{-1}d(\lambda,\mu;t).\end{aligned}$$

Using the continuity of the extended monotone metric K, $d(r\lambda,r\mu;t) = r^{-1}d(\lambda,\mu;t)$ holds for an arbitrary positive real number r which satisfies $r\lambda+r\mu<t$, and we can define $d(\lambda,\mu;t) := rd(r\lambda,r\mu;t)$ for $\lambda,\mu,t\in\mathbb{R}_+$ such that $\lambda+\mu\geqslant t$ by an arbitrary real number $r\in\mathbb{R}_+$ such that $r\lambda+r\mu<t$.

Set $\rho_0^{(n)} = \frac{1}{n}I_n$. When we take unique functions $b, c : \mathbb{R}_+ \to \mathbb{R}$ as in Lemma 3.1 for K, c satisfies

$$\begin{aligned} c(t) &= \frac{1}{4}K_{t\rho_0^{(2)}}(E_{12}^{(2)} + E_{21}^{(2)},\ E_{12}^{(2)} + E_{21}^{(2)}) \\ &= \frac{1}{4}K_{t\rho_0^{(2)}}(E_{12}^{(2)}\ \ E_{12}^{(2)}) + \frac{1}{4}K_{t\rho_0^{(2)}}(E_{21}^{(2)}\ \ E_{21}^{(2)}) \\ &= \frac{1}{2}d\left(\frac{1}{2}t, \frac{1}{2}t; t\right) \\ &= t^{-1}d(1, 1; t). \end{aligned}$$

Thus we get

$$\|A\|_\rho^2 = b(Tr\rho)|TrA|^2 + d(1, 1; t)Tr(\rho^{-1}A^*A).$$

For $A = [A_{ij}] \in M_n(\mathbb{C})$, we set $A_1 = diag(A_{11}, \ldots, A_{nn})$ and $A_2 = A - A_1$. Then

$$\begin{aligned} \|A\|^2_{\rho=diag(\lambda_1,\ldots\lambda_n)} &= \|A_1 + A_2\|_\rho^2 \\ &= \|A_1\|_\rho^2 + \|A_2\|_\rho^2 \\ &= b(Tr\rho)|TrA|^2 + d(1, 1; Tr\rho)Tr(\rho^{-1}A_1{}^*A_1) + \sum_{i\neq j} d(\lambda_i, \lambda_j; Tr\rho)|A_{ij}|^2 \\ &= b(Tr\rho)|TrA|^2 + \sum d(\lambda_i, \lambda_j; Tr\rho)|A_{ij}|^2. \end{aligned}$$

Because $b(t) + t^{-1}d(1, 1; t) = \|\rho_0^{(2)}\|^2_{t\rho_0^{(2)}} > 0$, $tb(t) + d(1, 1; t) > 0$ holds. We proved the assertion in case $n \geqslant 2$. When $n = 1$, the assertion is clear by Lemma 3.1. $\square$

Lemma 3.3. *The following two conditions for a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ are equivalent.*

(a) $K^f : \bigcup_{n=1}^\infty (M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times H_n^+) \to \mathbb{C}$ is an extended monotone metric.
(b) f is operator monotone.

Proof. That (b) implies (a) is shown by the proof of Theorem 3 in [10]. We assume the condition (a) and show the condition (b). Set $\mathbf{K}_\rho := f(\mathbf{L}_\rho \mathbf{R}_\rho{}^{-1})\mathbf{R}_\rho$. Then the follwing equivalence holds for any TP-CP $\mathbf{T}$.

$$\begin{aligned} \text{The monotonicity of} K^f &\Leftrightarrow \mathbf{T}^*\mathbf{K}_{\mathbf{T}(\rho)}^{-1}\mathbf{T} \leqslant \mathbf{K}_\rho^{-1} \\ &\Leftrightarrow \left(\mathbf{K}_\rho^{\frac{1}{2}}\mathbf{T}^*\mathbf{K}_{\mathbf{T}(\rho)}^{-\frac{1}{2}}\right)\left(\mathbf{K}_\rho^{\frac{1}{2}}\mathbf{T}^*\mathbf{K}_{\mathbf{T}(\rho)}^{-\frac{1}{2}}\right)^* \leqslant \mathbf{I} \\ &\Leftrightarrow \left(\mathbf{K}_\rho^{\frac{1}{2}}\mathbf{T}^*\mathbf{K}_{\mathbf{T}(\rho)}^{-\frac{1}{2}}\right)^*\left(\mathbf{K}_\rho^{\frac{1}{2}}\mathbf{T}^*\mathbf{K}_{\mathbf{T}(\rho)}^{-\frac{1}{2}}\right) \leqslant \mathbf{I} \\ &\Leftrightarrow \mathbf{T}\mathbf{K}_\rho\mathbf{T}^* \leqslant \mathbf{K}_{\mathbf{T}(\rho)}. \end{aligned}$$

We set

$$\mathbf{T}_{2n} : M_{2n}(\mathbb{C}) \ni \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \frac{1}{2}\begin{bmatrix} A + D & 0 \\ 0 & A + D \end{bmatrix} \in M_{2n}(\mathbb{C}).$$

This mapping is TP-CP and Hermitian with respect to Hilbert–Schmidt inner product. We take $\rho_1, \rho_2 \in H_n^+$, $0 \leqslant \lambda \leqslant 1$, $A \in M_n(\mathbb{C})$ and set

$$\rho := \lambda\rho_1 + (1 - \lambda)\rho_2 \in H_n^+, \rho' := \begin{bmatrix} \lambda\rho_1 & 0 \\ 0 & (1 - \lambda)\rho_2 \end{bmatrix} \in H_{2n}^+, \widetilde{A} := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \in M_{2n}(\mathbb{C}).$$

Then

$$
\begin{aligned}
&\langle \widetilde{A}, \mathbf{T}_{2n}\mathbf{K}_{\rho'}\mathbf{T}_{2n}^*(\widetilde{A})\rangle_{HS} \\
&\quad = Tr\mathbf{T}_{2n}(\widetilde{A})\mathbf{K}_{\rho'}(\mathbf{T}_{2n}(\widetilde{A})) \\
&\quad = Tr\widetilde{A}\mathbf{K}_{\rho'}(\widetilde{A}) \\
&\quad = Tr\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}^* \begin{bmatrix} f(\mathbf{L}_{\lambda\rho_1}\mathbf{R}_{\lambda\rho_1}^{-1})\mathbf{R}_{\lambda\rho_1}(A) & 0 \\ 0 & f(\mathbf{L}_{(1-\lambda)\rho_2}\mathbf{R}_{(1-\lambda)\rho_2}^{-1})\mathbf{R}_{(1-\lambda)\rho_2}(A) \end{bmatrix} \\
&\quad = Tr\{A^*(\lambda f(\mathbf{L}_{\rho_1}\mathbf{R}_{\rho_1}^{-1})\mathbf{R}_{\rho_1} + (1-\lambda)f(\mathbf{L}_{\rho_2}\mathbf{R}_{\rho_2}^{-1})\mathbf{R}_{\rho_2})(A)\} \\
&\quad = \langle A, (\lambda\mathbf{K}_{\rho_1} + (1-\lambda)\mathbf{K}_{\rho_2})(A)\rangle_{HS},
\end{aligned}
$$

$$
\begin{aligned}
\langle \widetilde{A}, \mathbf{K}_{\mathbf{T}_{2n}(\rho')}(\widetilde{A})\rangle_{HS} &= Tr\widetilde{A}^*\mathbf{K}_{\mathbf{T}_{2n}(\rho')}(\widetilde{A}) \\
&= Tr\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}^* \begin{bmatrix} \frac{1}{2}f(\mathbf{L}_{\rho}\mathbf{R}_{\rho}^{-1})\mathbf{R}_{\rho}(A) & 0 \\ 0 & \frac{1}{2}f(\mathbf{L}_{\rho}\mathbf{R}_{\rho}^{-1})\mathbf{R}_{\rho}(A) \end{bmatrix} \\
&= \langle A, \mathbf{K}_{\lambda\rho_1+(1-\lambda)\rho_2}(A)\rangle_{HS}.
\end{aligned}
$$

Since $\mathbf{T}_{2n}\mathbf{K}_{\rho'}\mathbf{T}_{2n}^* \leqslant \mathbf{K}_{\mathbf{T}_{2n}(\rho')}$, $\langle A, (\lambda\mathbf{K}_{\rho_1} + (1-\lambda)\mathbf{K}_{\rho_2})(A)\rangle_{HS} \leqslant \langle A, \mathbf{K}_{\lambda\rho_1+(1-\lambda)\rho_2}(A)\rangle_{HS}$. Therefore, a map $\rho \mapsto \mathbf{K}_{\rho}$ is concave:

$$\lambda\mathbf{K}_{\rho_1} + (1-\lambda)\mathbf{K}_{\rho_2} \leqslant \mathbf{K}_{\lambda\rho_1+(1-\lambda)\rho_2}.$$

We can show the operator concavity of f by using the concavity of $\rho \mapsto \mathbf{K}_{\rho}$. We take $\rho_1, \rho_2 \in H_n^+$, $0 \leqslant \lambda \leq 1, A \in M_n(\mathbb{C})$ again and set

$$\rho_i'' := \begin{bmatrix} \rho_i & 0 \\ 0 & I \end{bmatrix} (i = 1, 2), \rho := \lambda\rho_1 + (1-\lambda)\rho_2, \rho'' := \lambda\rho_1'' + (1-\lambda)\rho_2''.$$

Then

$$
\begin{aligned}
\left\langle \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}, \mathbf{K}_{\rho''}\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right)\right\rangle_{HS} &= Tr\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 0 & f(\mathbf{L}_{\rho})(A) \\ 0 & 0 \end{bmatrix} \\
&= TrA^*f(\rho)A,
\end{aligned}
$$

$$\left\langle \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}, (\lambda\mathbf{K}_{\rho_1''} + (1-\lambda)\mathbf{K}_{\rho_2''})\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right)\right\rangle_{HS} = TrA^*(\lambda f(\rho_1) + (1-\lambda)f(\rho_2))A.$$

Since the concavity of $\rho \mapsto \mathbf{K}_{\rho}$,

$$
\begin{aligned}
TrA^*f(\lambda\rho_1 + (1-\lambda)\rho_2)A &= \left\langle \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}, \mathbf{K}_{\rho''}\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right)\right\rangle_{HS} \\
&\geqslant \left\langle \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}, (\lambda\mathbf{K}_{\rho_1''} + (1-\lambda)\mathbf{K}_{\rho_2''})\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right)\right\rangle_{HS} \\
&= TrA^*(\lambda f(\rho_1) + (1-\lambda)f(\rho_2))A.
\end{aligned}
$$

Therefore, f is operator concave:

$$f(\lambda\rho_1 + (1-\lambda)\rho_2) \geqslant \lambda f(\rho_1) + (1-\lambda)f(\rho_2).$$

For a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the operator concavity is equivalent to the operator monotonicity (see [7]). $\square$

Now we shall prove the main theorem.

Proof of Theorem 3.2. Let K be a mapping derived from a pair of a continuous function b and a continuous family of operator monotone functions $\{f_t\}_{t\in\mathbb{R}_+}$ such that hold the condition of the statement as follows

$$K_\rho(A,B) := b(Tr\rho)(TrA)^*(TrB) + K_\rho^{f_{Tr\rho}}(A,B).$$

Our first goal is to show K is an extended monotone metric. it is obvious that K holds the sesquilinearness and the continuity. First, we will show the monotonicity of K. It is enough to show K^f to be an extended monotone metric. Since K^f has the monotonicity for $\rho \in H_n^+$ with trace 1 [9],

$$K_\rho^{f_{Tr\rho}}(A,A) = (Tr\rho)^{-1} K_{\frac{\rho}{Tr\rho}}^{f_{Tr\rho}}(A,A) \geqslant (Tr\rho)^{-1} K_{\mathbf{T}\left(\frac{\rho}{Tr\rho}\right)}^{f_{Tr\mathbf{T}(\rho)}}(\mathbf{T}(A),\mathbf{T}(A)) = K_{\mathbf{T}(\rho)}^{f_{Tr\mathbf{T}(\rho)}}(\mathbf{T}(A),\mathbf{T}(A))$$

for any $\rho \in H_n^+$, and TP-CP mappings $\mathbf{T}$.

Secondly, we show the positive definiteness of K. K has the unitary invariance, so we can regard $\rho \in H_n^+$ as diagonal.

$$\begin{aligned}
K_{diag(\lambda_1,\ldots,\lambda_n)}(A,A) &= b(Tr\rho)|TrA|^2 + K^{f_{Tr\rho}}{}_\rho(A,A) \\
&= b(Tr\rho)|TrA|^2 + \langle A, (f_{Tr\rho}(\mathbf{L}_\rho\mathbf{R}_\rho^{-1})\mathbf{R}_\rho)^{-1}A\rangle_{HS} \\
&= b(Tr\rho)|TrA|^2 + \sum \bar{A}_{ij}A_{kl}\left\langle E_{ij}^{(n)}, \left(f_{Tr\rho}\left(\frac{\lambda_k}{\lambda_l}\right)\lambda_l\right)^{-1} E_{kl}^{(n)}\right\rangle_{HS} \\
&= b(Tr\rho)|TrA|^2 + \sum |A_{ij}|^2 \left(f_{Tr\rho}\left(\frac{\lambda_k}{\lambda_l}\right)\lambda_l\right)^{-1} \\
&\geqslant -((Tr\rho)f_{Tr\rho}(1))^{-1}|TrA|^2 + \sum |A_{ij}|^2 \left(f_{Tr\rho}\left(\frac{\lambda_i}{\lambda_j}\right)\lambda_j\right)^{-1} \\
&= ((Tr\rho)f_{Tr\rho}(1))^{-1}\left(\left(\sum \lambda_j\right)\left(\sum \lambda_i{}^{-1}|A_{ii}|\right) - |\sum A_{ii}|^2\right) \\
&\quad + \sum_{i\neq j}\left(\lambda_j f_{Tr\rho}\left(\frac{\lambda_i}{\lambda_j}\right)\right)^{-1}|A_{ij}|^2 \\
&\geqslant \sum_{i\neq j}\left(\lambda_j f_{Tr\rho}\left(\frac{\lambda_i}{\lambda_j}\right)\right)^{-1}|A_{ij}|^2 \\
&\geqslant 0.
\end{aligned}$$

The second inequality is followed by the Schwartz inequality.

$K_{diag(\lambda_1,\ldots,\lambda_n)}(A,A) = 0$ holds if and only if the equality holds in the above three inequalities. The equality in the first inequality implies $|TrA| = 0$, and the equality in the second inequlity implies that there exists $r \in \mathbb{R}$ such that $A_{ii} = r\lambda_i$ for all i. Thus $0 = |TrA| = |r|\sum\lambda_i = |r|Tr\rho$. Since $r = 0$, we get $A_{ii} = 0$. The equality in the third inequality implies $A_{ij} = 0$ $(i \neq j)$.

Therefore K is an extended monotone metric.

Our second goal is to show that the mapping from b and $\{f_t\}$ to K is bijective. It is injective clearly. We see that for any monotone metric there exists a pair of b and $\{f_t\}$ which satisfies the desired condition. Fix a monotone metric K. We take a pair of b and d appear in Lemma 3.2 and set $f_t(x) := d(x,1;t)^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$. Since K^f has unitary invariance, we can regard $\rho \in H_n^+$ as diagonal. Then we get

$$\begin{aligned}
K_{\rho=diag(\lambda_1,\ldots,\lambda_n)}^{f_{Tr\rho}}(A,A) &= \langle A, (f_{Tr\rho}(\mathbf{L}_\rho\mathbf{R}_\rho{}^{-1})\mathbf{R}_\rho)^{-1}A\rangle_{HS} \\
&= \sum \bar{A}_{ij}A_{kl}\left\langle E_{ij}^{(n)}, \left(f_{Tr\rho}\left(\frac{\lambda_k}{\lambda_l}\right)\lambda_l\right)^{-1} E_{kl}^{(n)}\right\rangle_{HS} \\
&= \sum d(\lambda_i,\lambda_j;Tr\rho)|A_{ij}|^2 \\
&= \|A\|_\rho^2 - b(Tr\rho)|TrA|^2.
\end{aligned}$$

By the use of the polarization identity,

$$\begin{aligned}K_\rho^{f_{\mathrm{Tr}\rho}}(A,B)=&\frac{1}{4}\{K_\rho^{f_{\mathrm{Tr}\rho}}(A+B,A+B)-K_\rho^{f_{\mathrm{Tr}\rho}}(A-B,A-B)\\&+\sqrt{-1}K_\rho^{f_{\mathrm{Tr}\rho}}(A+\sqrt{-1}B,A+\sqrt{-1}B)-\sqrt{-1}K_\rho^{f_{\mathrm{Tr}\rho}}(A-\sqrt{-1}B,A-\sqrt{-1}B)\}\\=&K_\rho(A,B)-b(Tr\rho)(TrA)^*(TrB).\end{aligned}$$

Thus K^f is an extended monotone metric and f_t is operator monotone by Lamma 3.3. Since $b(t)+t^{-1}f_t(1)^{-1}=K_{\rho_0}(\rho_0,\rho_0)>0$, the mapping from b and $\{f_t\}$ to K is surjective. Therefore, we completed to prove this theorem. $\square$

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